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SEIFERT FIBERED SURGERIES WITH DISTINCT PRIMITIVE/SEIFERT POSITIONS

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ABSTRACT. We call a pair (K, m) of a knot K in the 3–sphere S^3 and an integer m a Seifert fibered surgery if m–surgery on K yields a Seifert fiber space. For most known Seifert fibered surgeries (K, m), K can be embedded in a genus 2 Heegaard surface of S^3 in a primitive/Seifert position, the concept introduced by Dean as a natural extension of primitive/primitive position defined by Berge. Recently Guntel has given an infinite family of Seifert fibered surgeries each of which has distinct primitive/Seifert positions. In this paper we give yet other infinite families of Seifert fibered surgeries with distinct primitive/Seifert positions from a different point of view.

1. INTRODUCTION

Let (K, m) be a pair of a knot K in S^3 and an integer m, and denote by $K(m)$ the manifold obtained from S^3 by m–surgery on K. We say that (K, m) is a *Seifert fibered surgery* if $K(m)$ is a Seifert fiber space. We regard that two Seifert fibered surgeries (K, m) and (K', m') are the *same* if K has the same knot type as K' (i.e. K is isotopic to K' in S^3) and $m = m'$. For a genus 2 handlebody H and a simple closed curve c in ∂H, we denote H with a 2–handle attached along c by $H[c]$.

Let $S^3 = V \cup_F W$ be a genus 2 Heegaard splitting of S^3, i.e. V and W are genus 2 handlebodies in S^3 with $V \cap W$ a genus 2 Heegaard surface F. It is known that such a splitting is unique up to isotopy in S^3 [17]. We say that a Seifert fibered surgery (K, m) has a *primitive/Seifert position* (F, K', m) if K' is a simple closed curve in a genus 2 Heegaard surface F such that $K'(\subset S^3)$ has the same knot type as K and satisfies the following three conditions.

- K' is *primitive* with respect to V, i.e. $V[K']$ is a solid torus.
- K' is *Seifert* with respect to W, i.e. $W[K']$ is a Seifert fiber space with the base orbifold $D^2(p, q)$ $(p, q \geq 2)$.
- The *surface slope* of K' with respect to F (i.e. the isotopy class in $\partial N(K')$ represented by a component of $\partial N(K') \cap F$) coincides with the surgery slope m.

For the primitive/Seifert position (F, K', m) above, we define the *index set* $i(F, K', m)$ to be the set $\{p, q\}$.

In general, if a knot K in S^3 has a primitive/Seifert position with surface slope m, then K is strongly invertible ([14, Claim 5.3]) and $K(m) \cong V[K] \cup W[K]$ is a Seifert fiber space or a connected sum of lens spaces. In particular, if K is hyperbolic, then the latter case cannot happen by the positive solution to the cabling conjecture for strongly invertible knots [4].

The notion of primitive/Seifert position was introduced by Dean [2] as a natural modification of Berge's primitive/primitive position [1]. It is conjectured that all the lens surgeries have primitive/primitive positions [7]. On the other hand, there are infinitely many Seifert fibered surgeries with no primitive/Seifert positions [11, 3, 16]; nevertheless the majority of Seifert fibered surgeries

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have such positions. Let (K, m) be a Seifert fibered surgery with two primitive/Seifert positions (F_1, K_1, m) and (F_2, K_2, m). Then, we say that (F_1, K_1, m) and (F_2, K_2, m) are the *same* if there is an orientation preserving homeomorphism f of S^3 such that $f(F_1) = F_2$ and $f(K_1) = K_2$; otherwise, they are *distinct*. It is natural to ask whether a Seifert fibered surgery (K, m) can have distinct primitive/Seifert positions. Recently Guntel [8] has given an infinite family of such examples. Her examples are twisted torus knots studied by Dean [2]. Among them, she finds infinitely many pairs of knots K_1, K_2 which have primitive/Seifert positions with the same surface slopes, and shows that K_1, K_2 are actually the same as knots in S^3, but their primitive/Seifert positions are distinct.

Theorem 1.1 ([8]). *There exist infinitely many Seifert fibered surgeries each of which has distinct primitive/Seifert positions.*

Remark 1.2. In Theorem 1.1, we can choose a Seifert fibered surgery (K, m) with distinct primitive/Seifert positions so that K is a hyperbolic knot whose complement $S^3 - K$ has an arbitrarily large volume.

In the present paper, we give yet other families of Seifert fibered surgeries with distinct primitive/Seifert positions from a different point of view. Our examples are twisted torus knots studied in [13, 14] (Theorem 2.1), and also Seifert fibered surgeries constructed by the Montesinos trick in [5, 6] (Theorem 3.3). We find infinitely many knots such that each knot K lies in two genus 2 Heegaard surfaces F_1, F_2 with the same surface slopes m, and (F_1, K, m) and (F_2, K, m) are distinct primitive/Seifert positions.

We use Lemma 1.3 to show that two primitive/Seifert positions are distinct.

Lemma 1.3. *Two primitive/Seifert positions (F_1, K_1, m) and (F_2, K_2, m) for a Seifert fibered surgery (K, m) are distinct if $i(F_1, K_1, m) \neq i(F_2, K_2, m)$.*

Proof of Lemma 1.3. Let us denote the Heegaard splitting of S^3 given by F_1 (resp. F_2) by $V \cup_{F_1} W$ (resp. $V' \cup_{F_2} W'$). We may assume that $V[K_1]$ (resp. $V'[K_2]$) is a solid torus, and $W[K_1]$ (resp. $W'[K_2]$) is a Seifert fiber space with the base orbifold $D^2(p, q)$ (resp. $D^2(p', q')$). Suppose for a contradiction that we have an orientation preserving homeomorphism f of S^3 such that $f(K_1) = K_2$ and $f(F_1) = F_2$. Then there are two cases to consider: $f(V) = V'$ or $f(V) = W'$. In the former case $f(W) = W'$ and we have also an orientation preserving homeomorphism $f_W : W[K_1] \to W'[K_2]$. This then implies that $\{p, q\} = \{p', q'\}$, i.e. $i(F_1, K_1, m) = i(F_2, K_2, m)$. This is a contradiction. In the latter $f(W) = V'$ and we have an orientation preserving homeomorphism $f_V : V[K_1] \to W'[K_2]$. However, this is impossible because $V[K_1]$ is a solid torus and $W'[K_2]$ is a Seifert fiber space over the base orbifold $D^2(p', q')$ $(p', q' \geq 2)$. □(Lemma 1.3)

2. Seifert fibered surgeries which have distinct primitive/Seifert positions I

Let V_1 be a standardly embedded solid torus in S^3; denote the solid torus $S^3 - \mathrm{int}V_1$ by V_2. Let $T_{p,q}$ be a torus knot which lies in ∂V_1 and wraps p times meridionally and q times longitudinally in V_1. Take a trivial knot $c_{p,q}$ in $S^3 - T_{p,q}$ as in Figure 1; $c_{p,q} \cap V_i$ consists of a single properly embedded arc in V_i which is parallel to ∂V_i. Note that the linking number $\mathrm{lk}(T_{p,q}, c_{p,q})$ with orientations indicated in Figure 1 is $p+q$, and that $c_{p,q}$ is a meridian of $T_{p,q}$ if $|p+q| = 1$. So in the following we assume $|p+q| > 1$. We denote by $K(p, q, p+q, n)$ the twisted torus knot obtained from $T_{p,q}$ by twisting n times along $c_{p,q}$. As shown in [12, Claim 9.2] ([3, Theorem 3.19(3)]), $T_{p,q} \cup c_{p,q}$ is a hyperbolic link in S^3. Hence by [3, Proposition 5.11] $K(p, q, p+q, n)$ is a hyperbolic knot if $|n| > 3$. In the following, for simplicity, we denote $c_{p,q}$ by c.

In [14] it is shown that $(pq + n(p+q)^2)$–surgery on $K(p, q, p+q, n)$ yields a Seifert fiber space over S^2 with at most three exceptional fibers of indices $|p|, |q|, |n|$. If $n = 0$, then it is a connected

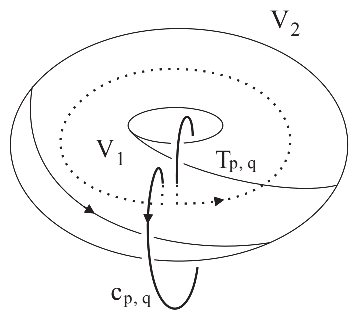


FIGURE 1.

sum of two lens spaces, if $n = \pm 1$, then it is a lens space. In fact, as shown in [3], $(K(p, q, p + q, \varepsilon), pq + \varepsilon(p + q)^2)$ is a Berge's lens surgery [1] of Type VII or VIII according as $\varepsilon = 1$ or -1.

Theorem 2.1. *Each Seifert fibered surgery* $(K(p, q, p + q, n),\ pq + n(p + q)^2)$ $(n \neq 0, \pm 1)$ *has distinct primitive/Seifert positions.*

The proof of Corollary 4.8 in [3] shows that for any r there are p and q such that for infinitely many n, $K(p, q, p+q, n)$ is a hyperbolic knot whose complement in S^3 has volume greater than r. Hence, Theorem 2.1 implies Theorem 1.1 and Remark 1.2.

Proof of Theorem 2.1. We follow the argument given in the proof of [14, Proposition 5.2]. Let us put $\tau_i = c \cap V_i$ $(i = 1, 2)$; $c = \tau_1 \cup \tau_2$. Then $H_1 = V_1 - \mathrm{int}N(c)$ and $H_2 = V_2 - \mathrm{int}N(c)$ are genus 2 handlebodies, and $T_{p,q}$ lies on $\partial V_i - \mathrm{int}N(c) = H_1 \cap H_2$. Note that $H_1 \cup H_2 = S^3 - \mathrm{int}N(c)$; see Figure 2.

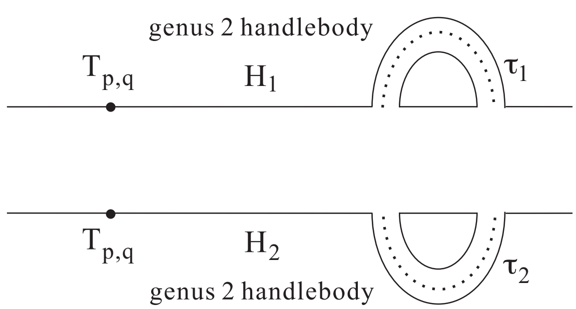


FIGURE 2.

We denote by U the solid torus glued to $S^3 - \mathrm{int}N(c)$ to construct the surgered manifold $c(-\frac{1}{n})$. Let $A_i = U \cap H_i$; $A_i(\subset \partial H_i)$ is an annulus whose core is a meridian of c. Since $\tau_i(\subset V_i)$ is parallel to ∂V_i, there is a disk Δ_i in H_i such that $\partial \Delta_i$ is the union of an arc in the annulus A_i and an arc in $\partial H_i - \mathrm{int}A_i$. Note that $N(\Delta_i) \cup U$ and the closure of $H_i - N(\Delta_i)$ are solid tori, and their intersection is a disk. This implies that $H_i \cup U = (N(\Delta_i) \cup U) \cup (H_i - N(\Delta_i))$ is a genus 2 handlebody for $i = 1, 2$.

Lemma 2.2. $(H_1 \cup U) \cup_F H_2$ *and* $H_1 \cup_{F'} (H_2 \cup U)$ *are both genus* 2 *Heegaard splitting of* $S^3 = c(-\frac{1}{n})$, *where* $F = \partial(H_1 \cup U) = \partial H_2$ *and* $F' = \partial(H_2 \cup U) = \partial H_1$.

Let $\{\mu, \lambda\}$ be a meridian-longitude basis for $H_1(\partial N(c))$. Then, a meridian and thus a longitude of U represent $-n\lambda + \mu$ and λ in $H_1(\partial N(c))$, respectively. It follows that a meridian of $N(c)$ winds U n times longitudinally. We thus have the following.

Lemma 2.3. *The core of the annulus $A_i (\subset \partial U)$ winds U n times longitudinally.*

The twisted torus knot $K(p, q, p + q, n)$ lies on F and F'. See Figure 3. In either case, the surface slope of $T_{p,q} = K(p, q, p+q, 0)$ is pq and the surface slope of $K(p, q, p+q, n)$ is the image of that of $T_{p,q}$ under n-twisting along c. Since $\mathrm{lk}(T_{p,q}, c) = p+q$, the surface slope of $K(p, q, p+q, n)$ is $pq + n(p + q)^2$.

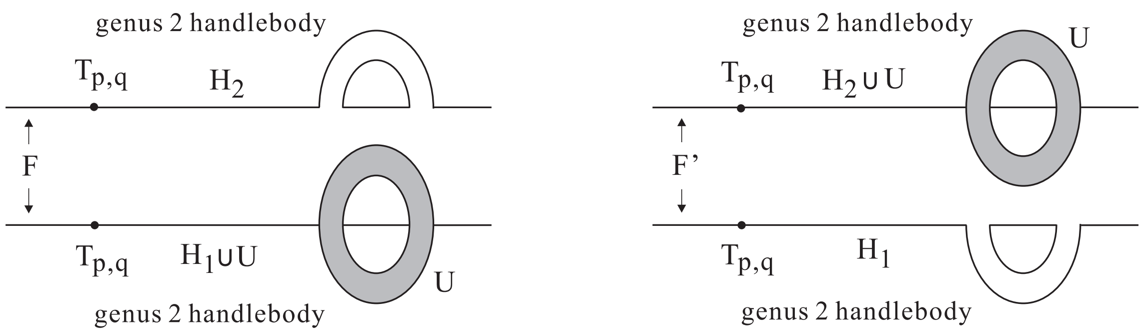


FIGURE 3. Genus 2 Heegaard splittings carrying $T_{p,q}$

Lemma 2.4. (1) *$H_1[T_{p,q}]$ is a fibered solid torus in which the core is an exceptional fiber of index $|q|$ and the core of A_1 is a regular fiber.*

(2) *$H_2[T_{p,q}]$ is a fibered solid torus in which the core is an exceptional fiber of index $|p|$ and the core of A_2 is a regular fiber.*

Proof of Lemma 2.4. See Lemma 9.1 in [12]. $\square$(Lemma 2.4)

Lemma 2.5. (1) *$(H_1 \cup U)[K(p, q, p+q, n)]$ is a Seifert fiber space over D^2 with two exceptional fibers of indices $|q|$, $|n|$.*

(2) *$(H_2 \cup U)[K(p, q, p + q, n)]$ is a Seifert fiber space over D^2 with two exceptional fibers of indices $|p|$, $|n|$.*

Proof of Lemma 2.5. First observe that $(H_1 \cup U)[K(p, q, p + q, n)] = H_1[T_{p,q}] \cup U$. Since a regular fiber of $H_1[T_{p,q}]$ contained in A_1 winds U n times longitudinally by Lemmas 2.3 and 2.4(1), $H_1[T_{p,q}] \cup U$ is a Seifert fiber space over D^2 with two exceptional fibers of indices $|q|, |n|$ as claimed in assertion (1). Assertion (2) follows in a similar fashion. $\square$(Lemma 2.5)

Therefore the Seifert fibered surgery $(K(p, q, p + q, n), pq + n(p + q)^2)$ has primitive/Seifert positions in two ways.

(1) $K(p, q, p + q, n)$ is primitive with respect to H_2 and Seifert with respect to $H_1 \cup U$; see Figure 4(i).

(2) $K(p, q, p + q, n)$ is primitive with respect to H_1 and Seifert with respect to $H_2 \cup U$; see Figure 4(ii).

To complete the proof of Theorem 2.1 let us show that $(F, K(p, q, p + q, n), pq + n(p + q)^2)$ and $(F', K(p, q, p+q, n), pq+n(p+q)^2)$ are distinct primitive/Seifert positions. The former has index $\{|q|, |n|\}$ by Lemma 2.5(1), and the latter has index $\{|p|, |n|\}$ by Lemma 2.5(2). Since p and q are relatively prime, $|p| \neq |q|$. Then, by Lemma 1.3 they are distinct. $\square$(Theorem 2.1)

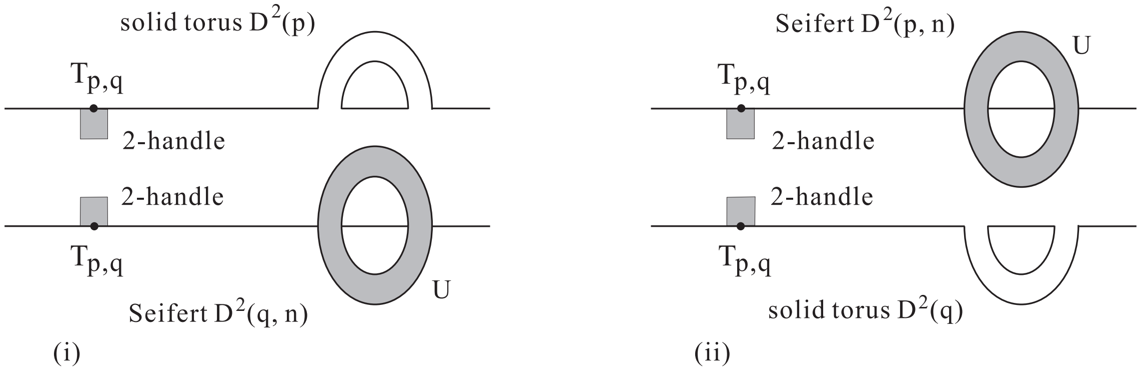


FIGURE 4.

Remark 2.6. Twisted torus knots $K(p, q, r, n)$ are obtained from torus knots $T_{p,q}$, roughly speaking, by twisting r strands of $T_{p,q}$ n times. In [2, Theorem 4.1], Dean obtains five classes of $K(p, q, r, \pm 1)$, where $p \geq 0, q \geq 0, 0 \leq r \leq p + q$, with primitive/Seifert positions. Let K be a knot in the classes. We can define H_1, H_2, U for K as for $K(p, q, p + q, n)$ above; then K is contained in two genus 2 Heegaard surfaces $F = \partial(H_1 \cup U), F' = \partial H_1$. Although $K(p, q, p + q, n)$ is primitive/Seifert with respect to both F and F', in general K is primitive/Seifert with respect to F only.

3. Seifert fibered surgeries which have distinct primitive/Seifert positions II

A *tangle* (B, t) is a pair of a 3–ball B and two disjoint arcs t properly embedded in B. A tangle (B, t) is a *rational tangle* if there is a pairwise homeomorphism from (B, t) to the trivial tangle $(D^2 \times [0, 1], \{x_1, x_2\} \times [0, 1])$ where D^2 is the unit disk and x_1 and x_2 are distinct points in $\mathrm{int}D^2$. Two rational tangles (B, t) and (B, t') are *equivalent* if there is a pairwise homeomorphism $h : (B, t) \to (B, t')$ such that $h|_{\partial B} = \mathrm{id}$. We can construct rational tangles from sequences of integers $[a_1, a_2, \ldots, a_n]$ as shown in Figure 5. Denote by $R(a_1, a_2, \ldots, a_n)$ the associated rational tangle. Each rational tangle can be parametrized by $r \in \mathbb{Q} \cup \{\infty\}$, where the rational number r is given by the continued fraction below. Thus we denote the rational tangle corresponding to r by $R(r)$.

$$r = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_1}}}$$

Let us consider the tangle $\mathcal{B}(A, B, C)$ given by Figure 6. In [6], the same tangle $\mathcal{B}(A, B, C)$ is defined by Figure 9(a) in [6]. However, the figure contains errors; four crossings of Figure 9(a) in [6] should be reversed. Figure 6 is the corrected diagram. The union of the tangle $\mathcal{B}(A, B, C) = (B_1, t_1)$ and a rational tangle $R(s) = (B_2, t_2)$ gives a pair $(S^3, \tau_s) = (B_1 \cup B_2, t_1 \cup t_2)$. We obtain τ_s, a knot or a link in S^3. In Figure 6 we illustrate the union of $\mathcal{B}(A, B, C)$ and $R(\infty)$.

In the following, we assume that τ_∞ is a trivial knot in S^3. Let $\pi_s : \widetilde{S^3}(s) \to S^3$ be the two-fold branched covering of S^3 along τ_s. Since τ_∞ is trivial, $\widetilde{S^3}(\infty) = S^3$. For a subset X of S^3, we often denote $\pi_s^{-1}(X)$ by $\widetilde{X}(s)$, and $\widetilde{X}(\infty)$ by $\widetilde{X}$ for simplicity. Let κ be an arc connecting the two vertical strings of $R(\infty)$ as the horizontal arc in Figure 6. Then the preimage $\pi_\infty^{-1}(\kappa)$ is a knot in S^3; we denote $\pi_\infty^{-1}(\kappa)$ by $k(A, B, C)$. Since the two-fold branched covering of B_2 along the rational

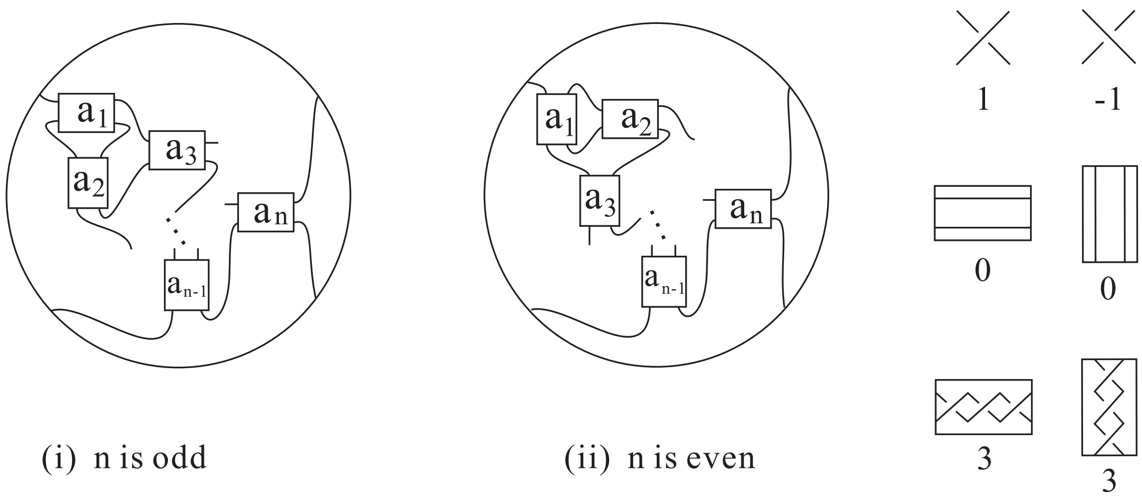
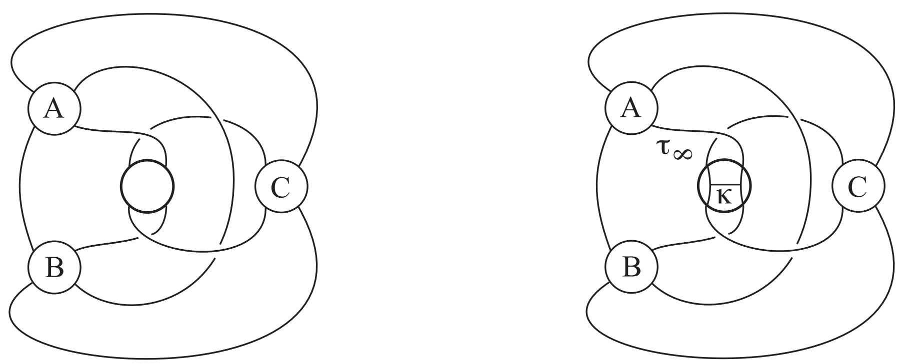


FIGURE 5. Rational tangles



$\mathcal{B}$(A, B, C) Union of $\mathcal{B}$(A, B, C) and R(∞)

FIGURE 6.

tangle t_2 is a solid torus, $\widetilde{B_2}(s)$ is a solid torus and in particular $\widetilde{B_2}$ is a tubular neighborhood of $k(A, B, C)$ in $\widetilde{S^3} = S^3$. Hence $\widetilde{S^3}(s)$ is obtained from S^3 by a Dehn surgery on $k(A, B, C)$. We denote the surgery slope by γ_s. For $(B_2, t_2) = R(s)$, if a properly embedded disk D in $B_2 - t_2$ separates the components of t_2, then $\widetilde{D}(s) = \pi_s^{-1}(D)$ consists of two meridian disks of the glued solid torus $\widetilde{B_2}(s)$. Hence, a component of $\partial\widetilde{D}(s) = \partial\widetilde{D}$ in $\partial\widetilde{B_2}$ represents the surgery slope γ_s.

Although Figure 9(a) in [6] contains errors as mentioned above, Lemma 5.1 in [6] is correct and we have:

Lemma 3.1 (Lemma 5.1 in [6]). *τ_∞ is a trivial knot in S^3 if either* (1) *or* (2) *below holds, where l, m, n, p are integers. The solutions are the only ones, up to interchanging A and B; note that there is a rotation interchanging them.*

(1) $A = R(l), B = R(m, -l), C = R(-n, 2, m-1, 2, 0)$
(2) $A = R(l), B = R(p, -2, m, -l), C = R(m-1, 2, 0)$

In case (1), we denote $k(A,B,C)$ by $k(l,m,n,0)$. In case (2), we denote $k(A,B,C)$ by $k(l,m,0,p)$. As shown in [5, 6], $k(A,B,C)$ are mostly hyperbolic knots. See [5, 6] for details.

The links τ_0 and τ_1 are Montesinos links with three branches indicated by the 3–balls B_A, B_B, B_C in Figures 7 and 8. Hence, $\widetilde{S^3}(s) = k(A,B,C)(\gamma_s)$, where $s = 0, 1$, is a Seifert fiber space whose exceptional fibers are the cores of $\widetilde{B_A}(s), \widetilde{B_B}(s), \widetilde{B_C}(s)$. Compute the rational numbers corresponding to the rational tangles $(B_A, B_A \cap \tau_s), (B_B, B_B \cap \tau_s), (B_C, B_C \cap \tau_s)$ such that A, B, and C satisfy (1) or (2) in Lemma 3.1; then, we obtain the indices of exceptional fibers of $k(A,B,C)(\gamma_s)$ as follows. If $(B_X, B_X \cap \tau_s)$ where $X \in \{A, B, C\}$ corresponds to a rational number $\frac{p}{q}$, then the Seifert invariant of the core of $\widetilde{B_X}(s)$ is $-\frac{q}{p}$, and the index is $|p|$.

Lemma 3.2 (corrected Proposition 5.4(2), (3), (5), (6) in [6])**.**

(1) (i) *γ_0–surgery on $k(l,m,n,0)$ produces a Seifert fiber space over S^2 with three exceptional fibers, the cores of $\widetilde{B_A}(0), \widetilde{B_B}(0), \widetilde{B_C}(0)$, of indices $|l-1|, |lm+m-1|, |2mn-m-n+1|$.*

(ii) *γ_1–surgery on $k(l,m,n,0)$ produces a Seifert fiber space over S^2 with three exceptional fibers, the cores of $\widetilde{B_A}(1), \widetilde{B_B}(1), \widetilde{B_C}(1)$, of indices $|l+1|, |lm-m-1|, |2mn-m+n|$.*

(2) (i) *γ_0–surgery on $k(l,m,0,p)$ produces a Seifert fiber space over S^2 with three exceptional fibers, the cores of $\widetilde{B_A}(0), \widetilde{B_B}(0), \widetilde{B_C}(0)$, of indices $|l-1|, |2lmp-lm-lp+2mp-m-3p+1|, |m-1|$.*

(ii) *γ_1–surgery on $k(l,m,0,p)$ produces a Seifert fiber space over S^2 with three exceptional fibers, the cores of $\widetilde{B_A}(1), \widetilde{B_B}(1), \widetilde{B_C}(1)$, of indices $|l+1|, |2lmp-lm-lp-2mp+m-p+1|, |m|$.*

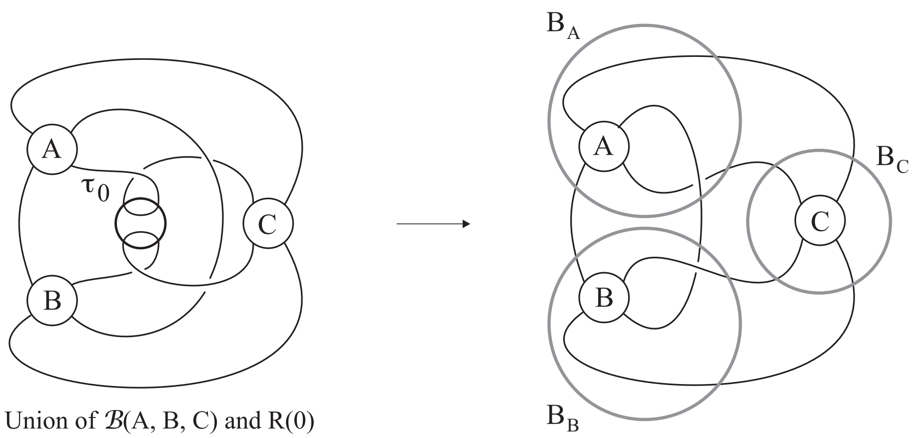


Figure 7. $\mathcal{B}(A,B,C) \cup R(0)$

In [6] a method is given to find primitive/Seifert positions for Seifert fibered surgeries constructed via tangles and double branched covers, and this is used to show that each of the surgeries $(k(A,B,C), \gamma_s)$ $(s = 0, 1)$ has a primitive/Seifert position. Using this method, we prove that each Seifert fibered surgery $(k(A,B,C),\ \gamma_s)$ $(s = 0, 1)$ has distinct primitive/Seifert positions if the indices of the exceptional fibers which are the cores of $\widetilde{B_A}(s)$ and $\widetilde{B_B}(s)$ are not equal.

Theorem 3.3. *According as A, B, C satisfy* (1) *or* (2) *of Lemma 3.1 we assume the following. If A, B, C satisfy Lemma 3.1*(1)*, assume that $|l-1| \neq |lm+m-1|$ if $s = 0$, and that $|l+1| \neq$*

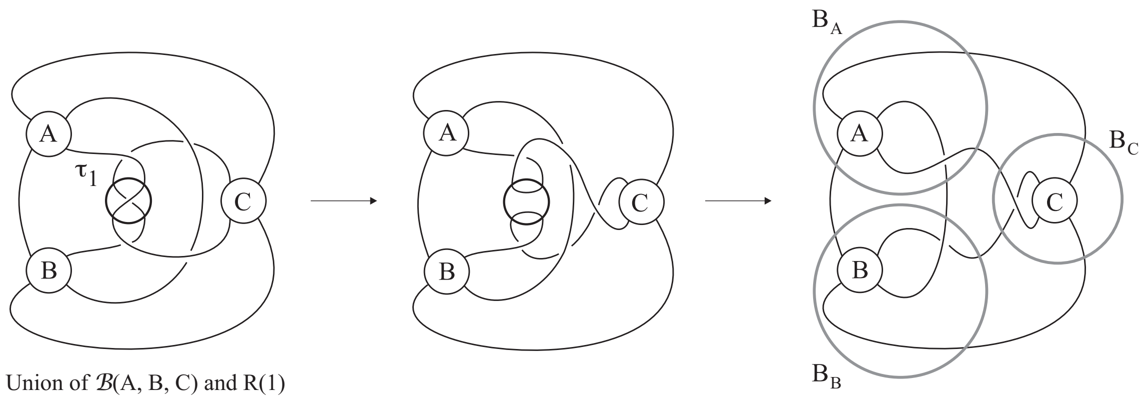


FIGURE 8. $\mathcal{B}(A,B,C) \cup R(1)$

$|lm-m-1|$ if $s=1$.
If A, B, C satisfy Lemma 3.1(2), assume that $|l-1| \neq |2lmp-lm-lp+2mp-m-3p+1|$ if $s=0$, and that $|l+1| \neq |2lmp-lm-lp-2mp+m-p+1|$ if $s=1$.
Then, each Seifert fibered surgery $(k(A,B,C),\gamma_s)$ $(s=0,1)$ has distinct primitive/Seifert positions.

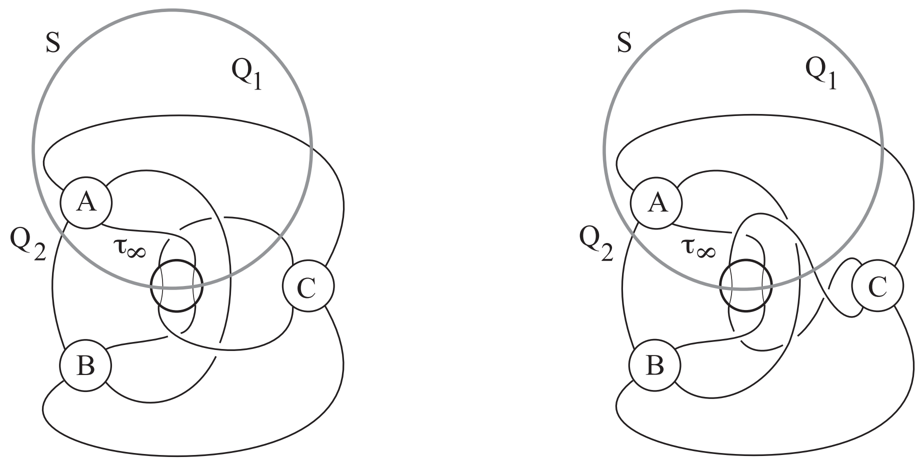
It follows from [6, Proposition 5.6] that the braid index of $k(l,m,n,0)$ is $2lm-1$ (resp. $2|lm|+1$) if $l>0, m>0$ (resp. $l>0, m<0$), and that of $k(l,m,0,p)$ is $2lm-l-1$ (resp. $2|lm|+l+1$) if $l>0, m>0$ (resp. $l>0, m<0$). Hence there are infinitely many knots satisfying the conditions in Theorem 3.3.

The assumption in Theorem 3.3 that the indices of the exceptional fibers in $\widetilde{B_A}(s)$ and $\widetilde{B_B}(s)$ are not equal is not a necessary condition for $(k(A,B,C),\gamma_s)$ to have distinct primitive/Seifert positions. Refer to Section 4.

Proof of Theorem 3.3. Let s be 0 or 1. If $s=0$, let S be the 2–sphere in S^3 shown in Figure 9(i), and if $s=1$, let S be the 2–sphere in S^3 shown in Figure 9(ii). In either case let Q_i $(i=1,2)$ be the 3–balls bounded by S as in Figure 9. Note that Figure 9 also describes the union of the tangles $\mathcal{B}(A,B,C)=(B_1,t_1)$ and $R(\infty)=(B_2,t_2)$, and $t_1 \cup t_2 = \tau_\infty$. However, τ_∞ in Figure 9(ii) is obtained by turning back a portion of τ_∞ in Figure 6. The tangles $(Q_i, Q_i \cap \tau_\infty)$ $(i=1,2)$ are 3–string trivial tangles. Hence, the two-fold branched covering $\widetilde{Q_1} \cup \widetilde{Q_2}$ gives a genus 2 Heegaard splitting of $S^3 = \widetilde{S^3}$, and $\widetilde{S} = \widetilde{Q_1} \cap \widetilde{Q_2}$ is a genus 2 Heegaard surface. Note that $S \cap B_2$ is a disk intersecting t_2 transversely in two points and containing the arc κ. This implies that the annulus $\widetilde{S \cap B_2}$ is a tubular neighborhood of the knot $k(A,B,C)=\widetilde{\kappa}$ in the Heegaard surface $\widetilde{S}$. Hence, a component of $\widetilde{S \cap \partial B_2}$ is a simple closed curve in $\partial\widetilde{B_2} = \partial N(k(A,B,C))$ representing the surface slope of $k(A,B,C)$ in the Heegaard surface $\widetilde{S}$.

Now let us show that the surface slope of $k(A,B,C)$ in $\widetilde{S}$ coincides with the surgery slope γ_s. Recall that γ_s–surgery on $k(A,B,C)$ corresponds to replacing $R(\infty)$ with $R(s)$. The disk $S \cap B_2$ in S is, as shown in Figure 9, a “horizontal” disk properly embedded in B_2. If $s=0$ and so S is as in Figure 9(i), then we may assume that $S \cap B_2$ separates the components of t_2 in $R(0)$ after replaced; see Figure 10. It follows that $\widetilde{S \cap \partial B_2}$ in $\partial\widetilde{B_2}$ represents the surgery slope γ_s, so that the surface slope of $k(A,B,C)$ in $\widetilde{S}$ coincides with γ_0 as desired. So assume that $s=1$ and S is as in Figure 9(ii). We need to see that the disk $S \cap B_2$ separates the components of t_2 in $R(1)$ attached to $\mathcal{B}(A,B,C)$ in Figures 6. The first isotopy in Figure 8 turns back a portion of τ_1. Then, in the second figure of Figure 8, t_2 in $R(1)$ becomes horizontal arcs. Hence we may assume that the

horizontal disk $S \cap B_2$ in Figure 9(ii) separates the components of t_2 in $R(1)$; see Figure 10. This implies that a component of $\widetilde{S \cap \partial B_2}$ also represents the surgery slope γ_1 as desired.



(i) (ii)

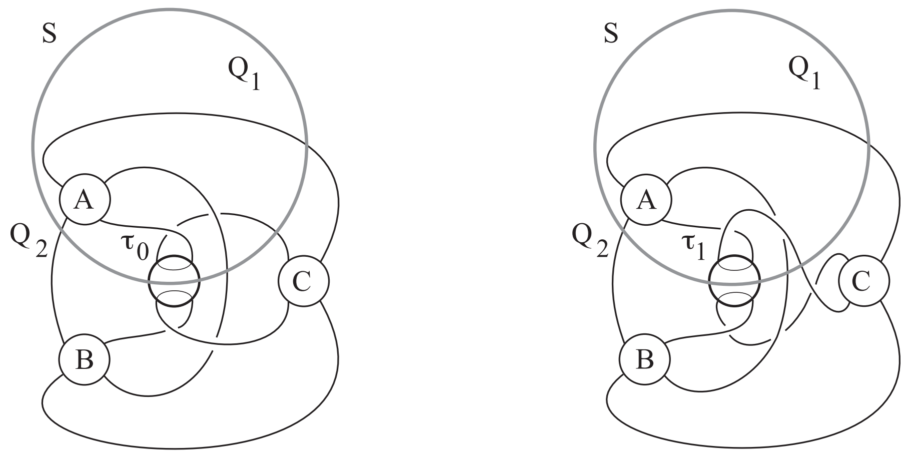
FIGURE 9. $\mathcal{B}(A, B, C) \cup R(\infty)$

Lemma 3.4. *The knot $K = k(A, B, C)$ is in a primitive/Seifert position in $\widetilde{S}$ with γ_s $(s = 0, 1)$ the surface slope, whose index set is the set of indices of exceptional fibers in $\widetilde{S^3}(s)$ corresponding to B_B, B_C.*

Proof of Lemma 3.4. We have already shown that the surface slope of K in $\widetilde{S}$ coincides with the surgery slope γ_s. We show that K is primitive with respect to the genus 2 handlebody $\widetilde{Q_1}$, and Seifert with respect to $\widetilde{Q_2}$. First consider $(S^3, \tau_s) = \mathcal{B}(A, B, C) \cup R(s)$. The 2–sphere S decomposes (S^3, τ_s) into two 2–string tangles $(Q_1, Q_1 \cap \tau_s)$ and $(Q_2, Q_2 \cap \tau_s)$; $(Q_1, Q_1 \cap \tau_s)$ is a rational tangle, and $(Q_2, Q_2 \cap \tau_s)$ is a partial sum of two rational tangles $(B_B, B_B \cap \tau_s)$ and $(B_C, B_C \cap \tau_s)$ in Figures 7, 8. This implies that $\widetilde{Q_2}(s)$ is a Seifert fiber space over the disk whose exceptional fibers are the cores of $\widetilde{B_B}(s), \widetilde{B_C}(s)$.

To complete the proof we prove $\widetilde{Q_i}(s) \cong \widetilde{Q_i}[K]$. We consider $(S^3, \tau_\infty) = \mathcal{B}(A, B, C) \cup R(\infty)$ again. The disk $Q_i \cap \partial B_1$ decomposes Q_i into two 3–balls $Q_i \cap B_1$ and $Q_i \cap B_2$, so that $\widetilde{Q_i}(s) = \widetilde{Q_i \cap B_1}(s) \cup \widetilde{Q_i \cap B_2}(s)$. Note that $B_1 \cap \tau_s = B_1 \cap \tau_\infty$, and τ_s intersects $Q_i \cap B_2$ in an arc whose end points lie in $Q_i \cap \partial B_1$. Hence, $\widetilde{Q_i \cap B_1}(s) = \widetilde{Q_i \cap B_1}$, and $\widetilde{Q_i \cap B_2}(s)$ is a 3–ball attached to $\widetilde{Q_i \cap B_1}$ along the annulus $\widetilde{Q_i \cap \partial B_1}$. In other words, $\widetilde{Q_i}(s)$ is obtained from $\widetilde{Q_i \cap B_1}$ by attaching a 2–handle along the annulus $\widetilde{Q_i \cap \partial B_1}$. Now replacing $R(s)$ with $R(\infty)$ again, let us see the relation between $\widetilde{Q_i}$ and $\widetilde{Q_i \cap B_1}$. It is not difficult to see the pairwise homeomorphism $(Q_i \cap B_2, Q_i \cap \partial B_2, Q_i \cap B_2 \cap \tau_\infty) \cong (D^2 \times [0, 1], D^2 \times \{1\}, \{x_1, x_2\} \times [0, 1])$, where $x_1, x_2 \in \mathrm{int} D^2$. This shows that $\widetilde{Q_i \cap \partial B_2}$ is a properly embedded annulus in $\widetilde{Q_i}$ parallel to $\widetilde{S \cap B_2}$, a tubular neighborhood of K in $\widetilde{S}$. Hence, there is a pairwise homeomorphism from $(\widetilde{Q_i \cap B_1}, \widetilde{Q_i \cap \partial B_1})$ to $(\widetilde{Q_i}, \widetilde{S \cap B_2})$. This implies $\widetilde{Q_i}(s) \cong \widetilde{Q_i}[K]$ as desired. □(Lemma 3.4)

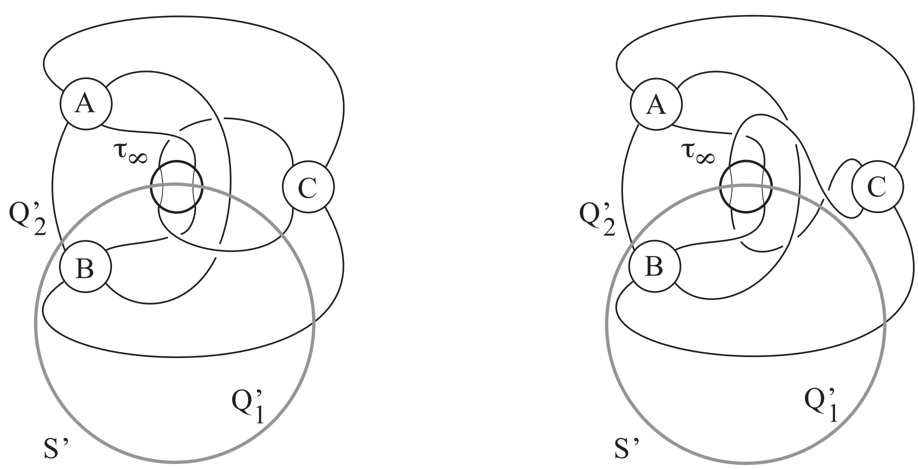
To find yet another primitive/Seifert position of $(k(A, B, C), \gamma_s)$, take the 2–sphere S' in S^3 as in (i) or (ii) of Figure 11 according as $s = 0$ or 1. Let Q'_i $(i = 1, 2)$ be the 3–balls bounded



Union of $\mathcal{B}$(A, B, C) and R(0)   Union of $\mathcal{B}$(A, B, C) and R(1)

FIGURE 10.

by S' as in Figure 11. Then, we can apply the arguments in the first and second paragraphs of the proof of Theorem 3.3 to S', Q'_1, Q'_2 instead of S, Q_1, Q_2. It follows that $\widetilde{Q'_1} \cup \widetilde{Q'_2}$ is a genus 2 Heegaard splitting of $\widetilde{S^3} = S^3$ with $\widetilde{S'}$ the Heegaard surface, and $k(A, B, C)$ is contained in $\widetilde{S'}$ with γ_s $(s = 0, 1)$ the surface slope.

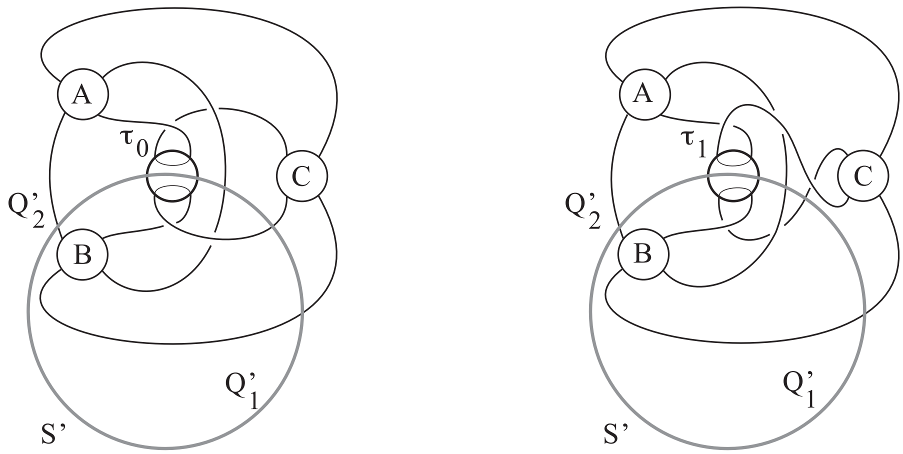


(i)   (ii)

FIGURE 11. $\mathcal{B}(A, B, C) \cup R(\infty)$

We can also apply most of the arguments in the proof of Lemma 3.4. The only difference is the fact $(Q'_2, Q'_2 \cap \tau_s)$ is a partial sum of $(B_A, B_A \cap \tau_s)$ and $(B_C, B_C \cap \tau_s)$ instead of $(B_B, B_B \cap \tau_s)$ and $(B_C, B_C \cap \tau_s)$; see Figure 12. Therefore, we obtain Lemma 3.5 below.

Lemma 3.5. *The knot $k(A,B,C)$ is in a primitive/Seifert position in $\widetilde{S}'$ with γ_s $(s=0,1)$ the surface slope, whose index set is the set of indices of exceptional fibers in $\widetilde{S^3}(s)$ corresponding to B_A, B_C.*



Union of $\mathcal{B}$(A, B, C) and R(0)   Union of $\mathcal{B}$(A, B, C) and R(1)

Figure 12.

Recall that by the assumption of Theorem 3.3 together with Lemma 3.2, the indices of the exceptional fibers of $\widetilde{S^3}(s) = k(A,B,C)(\gamma_s)$ corresponding to B_A and B_B are not equal. Lemmas 3.4 and 3.5 then imply that the index sets of the primitive/Seifert positions $(\widetilde{S}, k(A,B,C), \gamma_s)$ and $(\widetilde{S}', k(A,B,C), \gamma_s)$ are not equal. Hence, by Lemma 1.3 these are distinct primitive/Seifert positions for $(k(A,B,C), \gamma_s)$. This completes the proof of Theorem 3.3. $\square$(Theorem 3.3)

Remark 3.6. It follows from Lemma 3.2 that the set $\{k(A,B,C)(\gamma_s)\}$ $(s=0,1)$ consists of infinitely many Seifert fiber spaces. If two Seifert fibered surgeries $(k(A,B,C), \gamma_s)$ and $(k(A',B',C'), \gamma_s')$ are the same, then $k(A,B,C)(\gamma_s)$ and $k(A',B',C')(\gamma_s')$ are homeomorphic. Thus the set $\{(k(A,B,C), \gamma_s)\}$ contains infinitely many Seifert fibered surgeries.

4. Questions

In Theorems 2.1 and 3.3, Seifert fibered surgeries with distinct primitive/Seifert positions have distinct index sets. However, this is not always the case. Consider the Seifert fibered surgery $(k(2,4,n,0), \gamma_1)$ in Section 3. The result of γ_1–surgery on $K = k(2,4,n,0)$ is a Seifert fiber space with the base orbifold $S^2(\frac{-1}{3}, \frac{4}{3}, \frac{16n-7}{9n-4})$. Following Lemma 3.4, we see that (K, γ_1) has a primitive/Seifert position $(\widetilde{S}, K, \gamma_1)$ such that $\widetilde{Q_2}[K]$ is a Seifert fiber space over the disk with Seifert invariants $\frac{4}{3}, \frac{16n-7}{9n-4}$, where $\widetilde{Q_2}$ is a genus 2 handlebody bounded by $\widetilde{S}$. Similarly, Lemma 3.5 shows that (K, γ_1) has a primitive/Seifert position $(\widetilde{S}', K, \gamma_1)$ such that $\widetilde{Q_2'}[K]$ is a Seifert fiber space over the disk with Seifert invariants $\frac{-1}{3}, \frac{16n-7}{9n-4}$, where $\widetilde{Q_2'}$ is a genus 2 handlebody bounded by $\widetilde{S}'$. Thus $i(\widetilde{S}, K, \gamma_1) = i(\widetilde{S}', K, \gamma_1) = \{3, |9n-4|\}$. If $(\widetilde{S}, K, \gamma_1)$ and $(\widetilde{S}', K, \gamma_1)$ were the same, then following the argument in the proof of Lemma 1.3, we would have an orientation preserving homeomorphism from $\widetilde{Q_2}[K]$ to $\widetilde{Q_2'}[K]$; by [10, VI.18.Theorem] the homeomorphism is

fiber preserving up to isotopy. However, since $\frac{4}{3} \not\equiv \frac{-1}{3}$ mod 1, there is no such a homeomorphism [9, Proposition 2.1]. Hence the primitive/Seifert positions $(\widetilde{S}, k(2,4,n,0), \gamma_1)$ and $(\widetilde{S'}, k(2,4,n,0), \gamma_1)$ are distinct.

Question 4.1. Does there exist a Seifert fibered surgery which has distinct primitive/Seifert positions (F_1, K_1, m) and (F_2, K_2, m) satisfying the following condition?
Condition. Let W_i $(i = 1, 2)$ be a genus 2 handlebody bounded by F_i with respect to which $K_i(\subset \partial W_i)$ is Seifert. Then there is an orientation preserving homeomorphism from $W_1[K_1]$ to $W_2[K_2]$.

Even if a Seifert fibered surgery (K, m) has distinct primitive/Seifert positions, we expect that the number of such positions is not so large. In fact, we do not even have an example of a Seifert fibered surgery which has three primitive/Seifert positions.

Question 4.2. Does there exist a universal bound for the number of primitive/Seifert positions for a Seifert fibered surgery?

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